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WAVE PROPAGATION IN A TRUNCATED CONICAL SHELL

M. VALATHUR†

University of New Mexico, Albuquerque, New Mexico

Abstract—In this paper, the problem of wave propagation in a thin conical shell is considered. The material is assumed to be bilinear with rate-independent mechanical properties. A computational technique is proposed for solving the governing hyperbolic system of equations. This technique is a synthesis of a Laplace transform approach and the method of characteristics. Some numerical results are shown and the effect of geometrically induced non-homogeneity is discussed in terms of plastic wave attenuation.

INTRODUCTION

WAVE propagation in thin hollow cones has been studied intensively [1–3] in recent years. In Ref. [1], the problem of wave propagation in a thin hollow cone has been studied by a one-dimensional analytical representation. It has been concluded that, inclusion of second-order correction terms, such as lateral inertia, are necessary to improve the correlation between the theory and experiment.

In Ref. [2], the problem of impact of a thin conical shell was treated on the basis of membrane theory. A formal solution was obtained by a Fourier transformation and it was concluded that, if the forcing function could be properly filtered, the low frequency solution would be accurate. In Ref. [3], the problem of wave propagation in a thin hollow cone was formulated on the basis of membrane theory of thin shells. A formal solution was obtained by Laplace transformation, but numerical results could only be obtained by assymptotic methods for large values of the time after impact. The analogous problem of impact of a circular cylindrical shell was treated on the basis of membrane theory in Ref. [4]. The problem examined in the present paper relates to wave propagation in a thin hollow cone of an elastoplastic, bilinear material.

Based on the experiences in the case of the cylindrical shell [4] and the conical shell [3], the present paper also uses a Laplace transformation approach. The present analysis follows closely that of Berkowitz [4].

FORMULATION OF PROBLEM

Consider the problem of the impact of a thin, semi-infinite right circular conical shell. Based on the adequacy of membrane theory in the case of the cylindrical shell [4] and elastic right conical shell [2, 3], membrane theory will be used to formulate the present problem. If one considers the stress implied by the membrane formulation, it is seen that the forces exerted by the impact on the cone must at each point have the direction of the generatrix of the cone, a rather artificial assumption, the validity of which has been

† Now at: Pioneer Service and Engineering Co., 2 North Riverside Plaza, Chicago 60606.

discussed in Refs. [2, 3]. In a realistic situation, bending effects will occur in the vicinity of the edges of the shell and the present analysis cannot furnish these effects. But the membrane theory can be expected to be a good approximation elsewhere.

GOVERNING EQUATIONS

The equations of motion of the conical shell without bending (membrane theory) can be obtained by a simplification of the general equations for shells of revolution available in various forms in the literature [5, 6]. It is more convenient, however, to reformulate them in a form suitable for the present analysis.

The geometry of the middle surface of the shell of thickness $2h$ is shown in Fig. 1. The equations of motion are derived considering the equilibrium of the element $ABCD$ of the shell. Because of the axial symmetry, the forces N_r and N_θ are independent of θ. Furthermore, the element can undergo only two displacements: u_r in the direction of the generator and u_θ in the direction normal to the surface. These motions allow us to write the following two equations:

$$\rho\frac{\partial v_r}{\partial t} = \frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r} \tag{1}$$

$$\rho\frac{\partial v_\theta}{\partial t} = -\frac{\cot\alpha}{r}\sigma_\theta \tag{2}$$

where ρ is the mass per unit volume,

$$\sigma_r = \frac{N_r}{2h} \quad \text{and} \quad \sigma_\theta = \frac{N_\theta}{2h}$$

are the stresses acting on the faces of the element of the shell,

$$V_r = \frac{\partial u_r}{\partial t} \quad \text{and} \quad V = \frac{\partial u_\theta}{\partial t}$$

are the particle velocities.

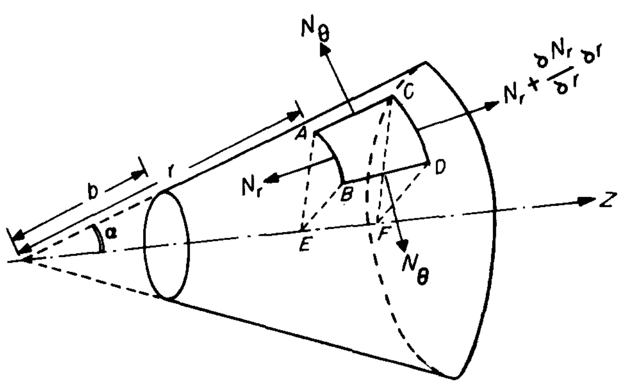

FIG. 1. Geometry and notation for the conical shell.

The strain–displacement relations are

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \tag{3}$$

$$\varepsilon_\theta = \frac{1}{r}(u_r + u_\theta \cot \alpha). \tag{4}$$

Hence the compatibility conditions are

$$\frac{\partial \varepsilon_r}{\partial t} = \frac{\partial v_r}{\partial r} \tag{5}$$

and

$$\frac{\partial \varepsilon_\theta}{\partial t} = \frac{1}{r}[v_r + v_\theta \cot \alpha]. \tag{6}$$

CONSTITUTIVE EQUATIONS

Assuming that the mechanical behaviour of the material under uniaxial stress condition can be represented by the piecewise-linear constitutive equation, we have

$$E\frac{\partial \varepsilon}{\partial t} = \frac{\partial \sigma}{\partial t}\begin{cases} \beta^2; & \sigma > \sigma_s \quad \text{and} \quad \dfrac{\partial \sigma}{\partial t} > 0 \\ 1; & \sigma < \sigma_s \quad \text{or} \quad \dfrac{\partial \sigma}{\partial t} < 0 \end{cases} \tag{7}$$

where σ_s is the yield stress and E is Young's modulus.

Further, yielding is assumed to occur at a state of stress governed by the von Mises yield condition, i.e.

$$(\sigma_e =)\sqrt{(\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta)} = \sigma_s. \tag{8}$$

Also, it is assumed that when a state of stress is "beyond" the yield surface, that is if the magnitude of the left side of equation (8) exceeds σ_s, then plastic flow can take place.

Based on the above assumptions, the following is a generalization of the one-dimensional law given by equation (7):

$$\begin{aligned} E\dot{\varepsilon}_r &= \begin{cases} (\beta^2-1)(\dot{\sigma}_r - \tfrac{1}{2}\dot{\sigma}_\theta) + (\dot{\sigma}_r - \mu\dot{\sigma}_\theta); & \text{plastic state and loading} \\ (\dot{\sigma}_r - \mu\dot{\sigma}_\theta); & \text{elastic state or unloading} \end{cases} \\ E\dot{\varepsilon}_\theta &= \begin{cases} (\beta^2-1)(\dot{\sigma}_\theta - \tfrac{1}{2}\dot{\sigma}_r) + (\dot{\sigma}_\theta - \mu\dot{\sigma}_r); & \text{plastic state and loading} \\ (\dot{\sigma}_\theta - \mu\dot{\sigma}_r); & \text{elastic or unloading} \end{cases} \end{aligned} \tag{9}$$

where μ is Poisson's ratio.

INITIAL AND BOUNDARY CONDITIONS

The initial and boundary conditions for the membrane after impact are

$$t = 0; \qquad r > b: v = \sigma = \varepsilon = 0 \tag{10}$$

$$\left.\begin{array}{l} t > 0; \qquad r = b: \sigma_r = \sigma_a \\ r \to \infty: v = \sigma = \varepsilon = 0 \end{array}\right\}. \tag{11}$$

It can be seen that the equations (1), (2), (5), (6) and (9) form a hyperbolic system and that the slope dr/dt of the characteristics in the (r, t) plane is either $\pm c_p$ or $\pm c_p/\beta$; where $c_p = \sqrt{[\frac{4}{3}(E/\rho)]}$. The first quadrant of (r, t) plane may be divided into the following regions (Fig. 2):

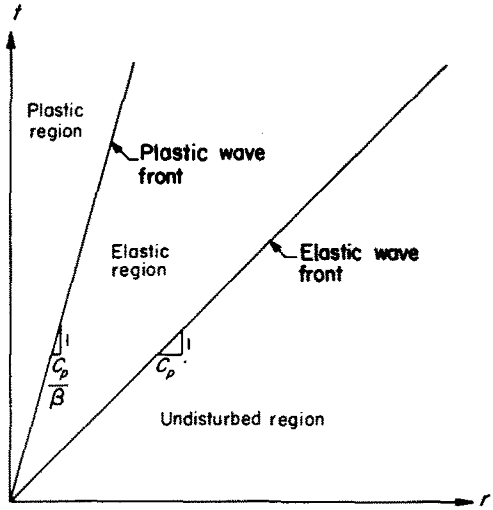

FIG. 2. $(r\text{–}t)$ plane diagram.

(i) $r > b + c_p t$: here $\sigma_r = \sigma_\theta = \varepsilon_r = \varepsilon_\theta = v_r = v_\theta = 0$; next,

(ii) $b + (c_p/\beta)t < r \leq b + c_p t: \sigma_e \leq \sigma_s$; so that the characteristic slopes are $\pm c_p$. If the impact is strong enough, just behind the line $r = (c_p/\beta)t$, the effective stress σ_e may reach σ_s, in such a case there is a region;

(iii) $b \leq r \leq b + (c_p/\beta)t$: a state of plastic loading and $\sigma_e > \sigma_s$, so that the characteristic slopes are $\pm c_p/\beta$.

SOLUTION BY LAPLACE TRANSFORM

Applying the Laplace transform with respect to time t to equations (1), (2), (5), (6) and (9) yields, after elimination of $\bar{\varepsilon}_r$, $\bar{\varepsilon}_\theta$, $\bar{\sigma}_\theta$ and $\bar{v}_\theta$,

$$\left.\begin{array}{r} \dfrac{E^*}{2}\rho r s^2 \bar{v}_r - \dfrac{\partial \bar{v}_r}{\partial r} + \dfrac{E^*}{2} s\bar{\sigma}_r - \dfrac{E^*}{2} r s \dfrac{\partial \bar{\sigma}_r}{\partial r} = 0 \\ \left[\dfrac{1}{rs} + \dfrac{\cot^2\alpha}{rs} + \dfrac{E^*}{2}\rho r s\right]\bar{v}_r - \left[\dfrac{E^*}{2} + \dfrac{\cot^2\alpha}{\rho r^2 s^2}\right]\bar{\sigma}_r - \left[\dfrac{E^*}{2} r + \dfrac{\cot^2\alpha}{\rho r s^2}\right]\dfrac{\partial \bar{\sigma}_r}{\partial r} = 0. \end{array}\right. \tag{12}$$

The transformed boundary conditions become

$$r = b, s > 0 : \bar{\sigma}_r = \frac{\sigma_a}{s} \tag{13}$$

where $E^* = \beta^2/E$, $\beta^2 = 1$ for elastic or unloading;

$\mu^* = 1/2E(2\mu+\beta^2-1)$, bars indicate the transformed quantities and s is the transform parameter.

Further it has been assumed that $\mu^*/E^* \simeq \frac{1}{2}$.

In the case of the cylindrical shell [4] and elastic right conical shell [3], it was found that the major effect behind the wave front was due to contributions from the inversion integral near $s = 0$. This suggests that an approximate solution of equations (12) for small values of s might lead to an approximate solution valid well behind the wave front. A similar approximation for large values of s would furnish a solution valid at the wave front. This procedure divides the effect into "low"—and "high"—frequency ones and determines them separately.

REGION I

From the physical point of view, this region is defined as an elastic region. The governing differential equations and the corresponding solutions have already been presented in Ref. [3] for this region. However, it will prove helpful to state here some results given earlier by Berkowitz and Bleich [3].

LOW-FREQUENCY EFFECTS VALID BEHIND THE WAVE FRONT

This approximation is obtained by simplification of equations (12), assuming that $s \to 0$ and also $rs \to 0$. Retaining only the lowest order term in s in the coefficients of each derivative in equations (12) leads to

$$r^2\frac{\partial^2\bar{\sigma}_r}{\partial r^2}+r\frac{\partial\bar{\sigma}_r}{\partial r}-\left[1+s^2\frac{\sec^2\alpha}{c_0^2}r^2\right]\bar{\sigma}_r = 0 \tag{14}$$

where $c_0 = \sqrt{(E/\rho)}$.

Solving the boundary value problem now posed by equation (14), results in

$$\bar{\sigma}_r(r,s) = \bar{\sigma}_r^{\text{I}}(r,s) = \frac{B}{s}\frac{K_1(sr/c_1)}{K_1(sb/c_1)} \tag{15}$$

where $c_1 = c_0 \cos\alpha$ and K_1 is modified Bessel function of order one.

Further $B = \sigma_s$ if $\sigma_a > \sigma_s$, otherwise $B = \sigma_a$. From inversion theorem,

$$\sigma_r(r,t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\bar{\sigma}_r\, e^{st}\, ds. \tag{16}$$

Further details require the inversion of expression (15). An approach by contour integration requiring extensive numerical work is outlined in Appendix. In this section, simple approximate expressions obtained by assymptotic approaches will be given.

Substituting equation (15) into (16) and using the method of steepest descent [7, 8], the value of σ_r becomes:

for $r-b \ll c_I t$,

$$\sigma_r(r, t) = B\frac{K_1(r/c_I t)}{K_1(b/c_I t)}. \tag{17}$$

To determine σ_r near the time of arrival, further simplification may be made by using the approximation valid for $s \to \infty$ in the integrand. Thus we have for $r-b \to c_I t^-$:

$$\sigma_r(r, t) \simeq B\left(\frac{b}{r}\right)^{\frac{1}{2}}\left[1+\frac{3}{8}c_I\left(\frac{1}{b}-\frac{1}{r}\right)\left(t-\frac{r-b}{c_I}\right)\right] \tag{18}$$

for $r-b < c_I t$; from Appendix,

$$\sigma_r(r, t) = \sigma_r^I(r, t) \simeq B(1-\Gamma) \tag{19}$$

where

$$\Gamma = \int_0^\infty \frac{I_1(rp/c_I)K_1(bp/c_I)-K_1(rp/c_I)I_1(bp/c_I)}{K_1^2(bp/c_I)+\pi^2 I_1^2(bp/c_I)}\frac{e^{pt}}{p}\,dp$$

where K_1 and I_1 are modified Bessel functions of order one.

HIGH FREQUENCY EFFECTS VALID AT THE WAVE FRONT

This approach is obtained by assuming $s \to \infty$ in equations (12) and simplifying the equations in a similar manner as for $s \to 0$. The resulting equation is

$$r^2\frac{\partial^2\bar{\sigma}_r}{\partial r^2}+r\frac{\partial\bar{\sigma}_r}{\partial r}-\left[1+\frac{r^2 s^2}{c_p^2}\right]\bar{\sigma}_r = 0 \tag{20}$$

solving the boundary value problem now posed by equation (20), results in

$$\bar{\sigma}_r(r, s) = \frac{B}{s}\frac{K_1(rs/c_p)}{K_1(bs/c_p)}. \tag{21}$$

However, equation (21) is based on approximate differential equation (20), in the derivation of which only the leading terms in s were retained. Consistent with this order of approximation, the expression

$$\frac{K_1(rs/c_p)}{K_1(bs/c_p)} \simeq \left(\frac{b}{r}\right)^{\frac{1}{2}}\exp\left\{-\frac{s}{c_p}(r-b)\right\}$$

is substituted into the inversion integral for σ_r. Evaluation by the method of steepest descent gives for $r-b = c_p t^-$:

$$\sigma_r \simeq B\left(\frac{b}{r}\right)^{\frac{1}{2}}. \tag{22}$$

REGION II

It can be easily shown that plastic waves propagate in this region. Further, it can be assumed here that the field variables can be expressed as the sum of elastic and plastic components,

$$\sigma = \sigma^e + \sigma^p$$
$$v = v^e + v^p.$$

LOW FREQUENCY EFFECTS VALID BEHIND THE WAVE FRONT

This approximation is obtained by assuming $s \to 0$ and also $sr \to 0$ in equations (12) and simplifying the equations in a similar manner as for region I,

$$r^2\frac{\partial^2\bar{\sigma}^p}{\partial r^2} + r\frac{\partial\bar{\sigma}^p}{\partial r} - [1 + s^2r^2\rho E^* \sec^2\alpha]\bar{\sigma}_r^p = 0. \tag{23}$$

Solving the boundary value problem, we have

$$\bar{\sigma}_r(r, s) = \frac{\sigma_0}{s}\frac{K_1(sr/c)}{K_1(sb/c)} + \bar{\sigma}_r^{\mathrm{I}} \tag{24}$$

where

$$c = \frac{c_0}{\beta}\cos\alpha$$

and $\bar{\sigma}_r^{\mathrm{I}}$ is given by equation (15).

From inversion theorem,

$$\sigma_r(r, t) = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \bar{\sigma}_r \, \mathrm{e}^{st} \, \mathrm{d}s. \tag{25}$$

From Appendix, we have for $r - b < ct^-$

$$\sigma_r(r, t) \simeq \sigma_0[1 - \Gamma] + \sigma_r^{\mathrm{I}} \tag{26}$$

where

$$\Gamma = \int_0^\infty \frac{I_1(rp/c)K_1(bp/c) - K_1(rp/c)I_1(br/c)}{K_1^2(bp/c) + \Pi^2 I_1^2(bp/c)}\frac{\mathrm{e}^{-pt}}{p}\,\mathrm{d}p.$$

To determine σ_r near the time of arrival of the plastic front, further simplification may be made by using the approximation valid for $s \to \infty$ in the integrand. Thus we have for $r - b \to ct^-$:

$$\sigma_r(r, t) \simeq \sigma_r^{\mathrm{I}} + \sigma_0\left(\frac{b}{r}\right)^{\frac{1}{2}}\left[1 + \frac{3}{8}c\left(\frac{1}{b} - \frac{1}{r}\right)\left(t - \frac{r - b}{c}\right)\right] \tag{27}$$

substituting equation (24) into (25) and using the method of steepest descent, the value of σ_r for $r - b \ll ct$ becomes

$$\sigma_r(r, t) \simeq \sigma_r^{\mathrm{I}} + \sigma_0\frac{K_1(r/ct)}{K_1(b/ct)}. \tag{28}$$

HIGH FREQUENCY EFFECTS VALID AT THE WAVE FRONT

This approach is obtained by assuming $s \to \infty$ in equations (12) and simplifying the equations in a similar manner as for region I. The resulting equation is

$$r^2 \frac{\partial^2 \bar{\sigma}^p}{\partial r^2} + \frac{r}{2} \frac{\partial \bar{\sigma}_r^p}{\partial r} - \left[\frac{1}{2} + s^2 \frac{r^2}{c_{\rm II}^2}\right] \bar{\sigma}_r^p = 0 \tag{29}$$

where

$$c_{\rm II} = \frac{c_p}{\beta}. \tag{30}$$

Also it has been assumed that $\mu^*/E^* \simeq \frac{1}{2}$.

Furthermore, the plastic wave front gets absorbed when σ_e [as given by equation (8)] on the wave front becomes equal to σ_s. Then a plastic wave of weak discontinuity $t = \Phi(r)$ starts propagating. The determination of this wave will be performed simultaneously with the solution in the entire region II on the basis of the relation $\sigma_e = \sigma_s$ on the wave $t = \Phi(r)$.

From [10], we have that $y = x^a \, \mathrm{e}^{-bx^d} Z_p(cx^s)$ is the solution of

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x[(1-2a) + 2\,\mathrm{d}bx^d] \frac{\mathrm{d}y}{\mathrm{d}x} + [a^2 - p^2 s^2 + s^2 c^2 x^{2s} - \mathrm{d}b(2a-d)x^d + \mathrm{d}^2 b^2 x^{2d}]y = 0.$$

If c is real, Z_p is to be interpreted by

$$Z_p = A_1 J_p(x) + A_2 J_{-p}(x)$$

or

$$Z_p = A_1 J_p(x) + A_2 Y_p(x),$$

where as if c is imaginary, Z_p stands for

$$Z_p = A_1 I_p(x) + A_2 I_{-p}(x)$$

or

$$Z_p = A_1 I_p(x) + A_2 K_p(x),$$

the choice in either case depends upon whether p is not or is zero or a positive integer, respectively. Comparing equation (29) with the above, we have

$$\bar{\sigma}_r(r, s) = r^{\frac{1}{4}} \left\{ A_1 I_{\frac{3}{4}}\left(\frac{sr}{c_{\rm II}}\right) + A_2 I_{-\frac{3}{4}}\left(\frac{sr}{c_{\rm II}}\right) \right\}. \tag{31}$$

But $\sigma_r(b, t) \neq \sigma_a$ and as $r \to \infty$ $\sigma_r = 0$

$$\bar{\sigma}_r(r, s) = \frac{(\sigma_a - \sigma_s)}{s} \left(\frac{r}{b}\right)^{\frac{1}{4}} \frac{I_{-\frac{3}{4}}(sr/c_{\rm II})}{I_{-\frac{3}{4}}(br/c_{\rm II})} + \bar{\sigma}_r^{\rm I}. \tag{32}$$

However, equation (32) is based on approximate differential equation (29), in the derivation of which only the leading terms in s were retained. Consistent with this order of approximation, the equation (32) can be written as

$$\bar{\sigma}_r(r, s) = \bar{\sigma}_r^{\rm I} + \frac{\sigma_0}{s} \left(\frac{b}{r}\right)^{\frac{1}{4}} \mathrm{e}^{-s/c_{\rm II}(r-b)} \tag{33}$$

where $\sigma_0 = \sigma_a - \sigma_s$.

Evaluation by the method of steepest descent gives, for $r-b = c_{II}t^-$

$$\sigma_r \simeq \sigma_0\left(\frac{b}{r}\right)^{\frac{1}{2}} + \sigma_r^I. \tag{34}$$

The value of $\bar{v}_r$ can be calculated from the second of equations (12). A similar approach gives the value of v_r:

for $r-b = c_{II}t^-$

$$V_r \simeq \frac{\sigma_0}{\rho c_{II}}\left(\frac{b}{r}\right)^{\frac{1}{2}} + \frac{1}{\rho c_I}\sigma_r^I. \tag{35}$$

Knowing σ_r and v_r, other quantities can be calculated.

NUMERICAL EXAMPLE AND DISCUSSION

Let us consider a numerical example illustrating the theoretical solutions of the previous section. The following data are assumed for the cone:

$E = 10{,}000{\cdot}0$ ksi;
$\mu = 0{\cdot}333$;
$\beta^2 = 80$;
yield stress $\sigma_s = 45$ ksi;
applied stress $\sigma_a = 55$ ksi;
$\rho = 2{\cdot}475 \times 10^{-7}$ kips-sec^2/in.^{4};
$\alpha = 9°$;
$b = 1{\cdot}58$ in.;
$2h = 0{\cdot}125$ in.

Complete calculations for each particular region of the (r, t) plane will not be given. The variation of σ_r with distance along the cone surface from vertex is shown in Fig. 3 at two instants of time.

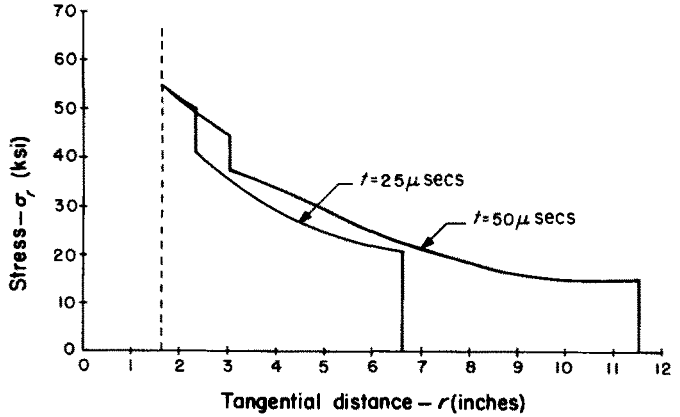

FIG. 3. Variation of stress with tangential distance at two values of time.

The numerical results presented in Fig. 3 indicate a rather complex pattern of elasto-plastic wave propagation in a thin conical shell. For the particular case treated in this paper, involving wave propagation into a bilinear material with rate independent mechanical properties, several interesting features can be noted. Under conditions of constant boundary stress, the geometrically induced non-homogeneity tends to initiate unloading behind the plastic precursor front. Further, the plastic front tends to become absorbed as it propagates into the material; this absorption can take place more rapidly for materials with low plastic moduli. When compared with the elastic precursor, the discontinuities in strain at the plastic front are generally more significant than the discontinuities in stress or particle velocity.

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REFERENCES

[1] V. H. KENNER, W. GOLDSMITH and J. L. SACKMAN, Longitudinal impact on a hollow cone. *J. appl. Mech.* **36**, 445–450 (1969).
[2] B. ALBRECHT, P. N. SONNENBERG and R. C. DOVE, On Elastic Wave Propagation in Truncated Conical Shells, Technical Report No. AFWL-TR-68-100, Air Force Base, New Mexico (1968).
[3] H. M. BERKOWITZ and H. H. BLEICH, Axial impact of an elastic right conical membrane shell. *AIAA Jnl* **4**, 1378–1384 (1966).
[4] H. M. BERKOWITZ, Longitudinal impact of a semi-infinite elastic cylindrical shell. *J. appl. Mech.* **30**, 347–354 (1963).
[5] W. FLUGGE, *Stresses in Shells*, pp. 37, 66. Springer-Verlag (1966).
[6] H. KRAUS, *Thin Elastic Shells*, pp. 94, 145. John Wiley (1967).
[7] H. JEFFREYS and B. S. JEFFREYS, *Methods of Mathematical Physics*, 3rd edition, pp. 462, 466. Cambridge University Press (1956).
[8] H. S. CARSLAW and J. C. JAEGER, *Operational Methods in Applied Mathematics*, 2nd edition, chapter 13. Oxford University Press (1953).
[9] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd edition, p. 92. Macmillan (1948).
[10] F. B. HILDEBRAND, *Advanced Calculus for Applications*, pp. 155, 156. Prentice-Hall (1965).

APPENDIX

The low-frequency approximation in the regions I and II leads to the question of inversion of the function

$$\bar{\sigma}_r(r, s) = \frac{1}{s}\frac{K_1(as)}{K_1(bs)}; \qquad a > b$$

$$\sigma_r(r, t) = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \bar{\sigma}_r(r, s)\, e^{st}\, ds$$

$$= \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \frac{1}{s}\frac{K_1(as)}{K_1(bs)}\, e^{st}\, ds. \tag{A.1}$$

The integrand of (A.1) has a branch point of $s = 0$ and so the contour of Fig. 4 must be used. It is easily verified that there are no poles in or on this contour and that the integrals over BF and AC tend to zero as $R \to \infty$.

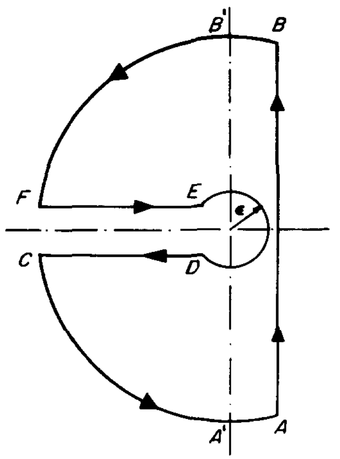

FIG. 4. Contour for transform inversion.

It follows that the above integral is equal to the sum of the integrals over CD and EF and that over the small circle, as $R \to \infty$ and $\varepsilon \to 0$.

In the integral along CD; $s = \rho\, e^{-i\pi}$

$$R \geq \rho \geq \varepsilon; \qquad R \to \infty; \qquad \varepsilon \to 0.$$

In the integral along EF, $s = \rho\, e^{i\pi}$

$$\varepsilon \leq \rho \leq R; \qquad R \to \infty; \qquad \varepsilon \to 0.$$

In the integral over the small circle, $s = \varepsilon e^{i\theta}$

$$|\theta| < \Pi - r; \qquad r \to 0, \qquad \varepsilon \to 0.$$

Also we know that

$$K_1(Z\, e^{m\Pi i}) = e^{-m\Pi i} K_1(Z) - \Pi i \frac{\sin m\Pi}{\sin \Pi} I_1(Z).$$

The integral over the small circle becomes $2\pi i$.

The integral along CD and EF becomes

$$2\Pi i \int_0^\infty \left[\frac{I_1(a\rho)K_1(b\rho) - K_1(a\rho)I_1(b\rho)}{K_1^2(b\rho) + \Pi^2 I_1(b\rho)}\right] \frac{e^{-\rho t}}{\rho}\, d\rho.$$

Hence we have

$$\sigma_r(r, t) = (1 - \Gamma) \tag{A.2}$$

where

$$\Gamma = \int_0^\infty \left[\frac{I_1(a\rho)K_1(b\rho) - K_1(a\rho)I_1(b\rho)}{K_1^2(b\rho) + \Pi^2 I_1(b\rho)}\right] \frac{e^{-\rho t}}{\rho}\, d\rho.$$

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Абстракт—В работе исследеется задача распространения волны в тонкой конической оболочке. Подразумевается, что материал является билинейным, механические свойства котораго не зависят от времени. Предлагается процесс расчета на вычислительной иашине для решения определяющей системы гиперболцческих уравнений. Зтот способ является синтезом подхода путем преобразования Лапласа и метода характеристик. Даются нокоторые численные резулвтаты. Обсуждаются зффекты геометрически введенной неоднородности в членах затухания пластической волны.